A CRASH COURSE IN GENERAL RELATIVITY AND GRAVITATIONAL WAVES

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1. Intro

This text is meant for those who have a good grasp on mathematics and physics, but who have not yet had experience with Gravitational Waves and General Relativity writ large. As a preliminary effort, it has several places of threadbare explanation, logical lacunae, and otherwise vague wording. It begins with a summary of how General Relativity accounts for precessional orbit of a planet orbiting a star, move on to an analytic model of a GW emitted by a binary merger's inspiral, and end with a brief explanation of how such mergers are detected.

2. Origins of GWs

2.1. **The Newtonian Model.**

In both Newtonian and modern conceptions, planets trace elliptical orbits around the Sun. This can be seen as a planet completing a circular orbit around the sun with angular frequency ω_ϕ, while also oscillating in its distance from the star with frequency ω_r. In the Newtonian model, these oscillations are perfectly in-sync, so that, when measuring from a common starting point, a planet left to repeat its orbit a thousand times will reach its furthest point from the star at the exact same angle, each time. This prediction arises from the fact that, in the Newtonian model, gravity is a force that acts instantly. For an example, in the Newtonian model, the moon "pulls" directly on the water in the ocean, and when the moon moves, the force on the water will change instantly to match the new position.

In this elliptical orbit, a planet is essentially in simple harmonic motion, where its energy shifts from more kinetic energy (at its closest point to the sun) to more potential energy (at the midpoint between the closest and furthest points, a distance which will now be considered r_0). This distance of minimum potential can be found from Newton's equation for potential energy per unit mass, which states that for a planet of mass m and angular momentum L orbiting a star of mass M,

$$\frac{U(r)}{m} = \frac{L^2}{2m^2r^2} - \frac{M}{r} \tag{1}$$

. Because we know that the planet is constrained in SHM, we can look for the value of r where $\frac{d(U(r)/m)}{dr} = 0$, secure in the knowledge that this value will be r_0, our distance of minimum potential,

$$r_0 = \frac{L^2}{Mm^2} \tag{2}$$

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rom here, we can obtain our orbital frequency, ω_ϕ by using another of Newton's useful derivations, namely that

$$\frac{L}{m} = r_0^2\omega_\phi \tag{3}$$

. This, in conjunction with (2), gives us the value

$$\omega_\phi = \frac{M}{r_0^3} \tag{4}$$

. For the elliptical oscillation, we can use the other well-known relation for SHM, that

$$\frac{U}{m} = \frac{\omega_r^2 x^2}{2} \tag{5}$$

and consequently that

$$\frac{d^2(U/m)}{dx^2} = \omega_r^2 \tag{6}$$

. Combining this with our description of U/m in (1), returns the expected frequency

$$\omega_r = \frac{M}{r_0^3} \tag{7}$$

. These relations tell us that, under Newton's model of gravity, $\omega_\phi = \omega_r$, QED.

2.2. **The Relativistic Model.** Einstein took offense to the idea that an object could impart a force on another object at a distance, without some sort of interstitial medium. In Einstein's model, also known as General Relativity, a massive body did not directly enact a force on its neighbors, but warped space around it in such a way that nearby objects would be accelerated towards the disturbance.

The General Relativistic equations of motion had a slightly different relation between potential and radius, using not the flat quantity U/m but the squared term $\frac{1}{2}(U/m)^2$. Because determining the frequency of the oscillation depends only on where the function $U(r)$ finds its minimum, we can for the most part disregard this new exponent. The GR model led to an equation similar to Newton's with two added terms:

$$\frac{1}{2}(\frac{U}{m})^2 = \frac{1}{2} + \frac{L^2}{2m^2r^2} - \frac{M}{r} - \frac{ML^2}{m^2r^3} \tag{8}$$

. Here, the addition of the r^-3 in the final term means that as the orbiting body approaches the center of the system, its potential energy wil approach zero, as the exponential growth of the negative final term will overwhelm the Newtonian $\frac{L^2}{2m^2r^2} - \frac{M}{r}$ terms. By taking the derivative of (8) with respect to r and setting it equal to zero, we can find a value

$$(\frac{L}{m})^2 = \frac{1}{1 - \frac{3M}{r}} \tag{9}$$

, which can then be used in the definition of angular momentum ($\frac{L}{m} = r_0(\omega_\phi)$) to say that

$$\omega_\phi^2 = \frac{M}{r_0^3}(1 - \frac{3M}{r_0})^{-1} \tag{10}$$

. Again, this is a value that is very similar to the Newtonian prediction, with the sole addition of a new coefficient.

For the radial frequency, we can substitute (9) into the second derivative of (8) with regards to r, which returns the value

$$\frac{d^2}{dr^2}(\frac{U}{2m})^2 = \omega_r^2 \approx \frac{M}{r_0^3}(1 - \frac{3M}{r_0}) \tag{11}$$

. We can clearly see, then, that Einstein's equation predicts a relation where the planet moves closer and further from the Sun slightly slower than it orbits, meaning that it never traces the same elliptical twice.

We can use the fact that the frequencies are close to each other to say that

$$\omega_\phi^2 - \omega_r^2 \approx 2\omega_\phi(\omega_\phi - \omega_r) \tag{12}$$

which, after substituting the values found in equations (10) and (11), yields the relation

$$\omega_\phi - \omega_r \approx \frac{3M}{r}\omega_\phi \tag{13}$$

. Which is an approximation for Einstein's prediction of how much the perihelion and aphelion of an elliptical orbit change over time.

2.3. **Consequences of Relativity, or The Post-Newtonian Model.** So far, we have covered what is called the "first-degree" Post-Newtonian, that is to say it is one degree of correction from Newton's original law of universal gravitation. As we progress further into the dark forest that is General Relativistic theory, we find that there are many such corrections proposed by Einstein's equations, with each additional term bringing the model slightly closer to perfect accuracy. Current cutting edge has seven orders of corrections, but we only need four to start seeing the effects of Gravitational Waves, the nominal topic of this paper.

As we mentioned above, General Relativity views gravity not as a force instantaneously exerted from one body to another, but a warping of space around large bodies of mass. A natural consequence of this perception is that, for particularly massive orbiting bodies, "pockets" of this warping can be created that move indefinitely far away from the orbiting binary that created them. These pockets are called Gravitational Waves, and can be predicted using the aforementioned Post-Newtonian corrections.

Generally, a Gravitational Wave propagating in the z direction will cause perturbations in the x and y directions. To visualize this imagine four stationary free-floating masses, a, b, c and d, at locations $a = (1, 0), b = (-1, 0), c = (0, 1), d = (0, -1)$, as an ideal sinusoidal Gravitational Wave passes through the plane at point $(0, 0)$. From our perspective, the distance between points a and b will stretch slightly, from $|ab| = 2$ to $|ab| = 2n$ for some real number n. At the same time, the distance cd will contract slightly, from $|cd| = 2$ to $|cd| = 2/n$, for the same n. Then, the process reverses, ending with $|ab| = 2/n$ and $|cd| = 2n$. From here, these two distances will continually oscillate between $2n$ and $2/n$, with one distance achieving its maximum just as the other achieves its minimum. In measured cases, n is very close to one, and the quantity $h = n - 1$ is called the strain, or amplitude, of the wave.

Of course, we can hardly expect every Gravitational Wave to line up clearly with our arbitrary axes in every situation. What if a hypothetical Gravitational Wave stretches and squishes not along $x = 0$ and $y = 0$, but along the lines $x = y$ and $x = -y$? Well, according to the logic we used above, it would be then lengths $|ac|$ and $|bd|$ that are stretched by n, and $|ad|$ and $|bc|$ that are contracted by n, and the distances $|ab|$ and $|cd|$ become $|ab| = |cd| = \sqrt{2(n^2 + n^{-2})}$ (notice how in the limit case $n >> 1$, we recover our initial case of $|ab| = |cd| = 2$).

These two limiting cases, called the h_+ and $h_\times$ polarizations of a wave, can be used to build any other orientation of Gravitational Wave, in the form

$$h_{tot} = Ah_+ - iBh_\times \tag{14}$$

, for some A and B, and the functions $h_+(t)$ and $h_\times(t)$.

3. Properties of GWs

3.1. **Measurable Waves.** Nearly all of the Gravitational Waves we measure today are caused by binary mergers, which is two neutron stars or black holes of comparable mass orbiting each other, emitting Gravitational Waves. Under the Newtonian model, these bodies would maintain an orbit infinitely (an object in motion...), but now we know that such a system is emitting Gravitational Waves, which means in turn that the system is losing energy over time. This means that two bodies in a stable orbit will eventually collapse and collide with each other, even in the absence of any external force.

In order to quantify these types of Gravitational Waves, we need to talk about some physical constants pertaining to the binaries that create them. First is the "symmetric mass ratio" of the orbiting bodies. For a body of mass m orbiting a body of mass M, this ratio is:

$$\eta = \frac{mM}{(m+M)^2} \tag{15}$$

. The other terms are much easier to explain, being the phase of the orbit Φ, the separation between the two orbiting bodies r, the distance from the point at which we are measuring $h(t)$ to the binary causing the waves, R, and the total mass of the binary system, M_o. With these, we can finally discuss the current model for the strength of gravity wave h, over time. Note that for this ideal case, we assume that the point we are measuring h from is in the plane in which the binary is orbiting, as the binary emits the strongest Gravitational Waves in that plane. With all of this said, the full equation for $h(t)$ for a point on the plane of orbit a distance R away from the binary's center of mass is:

$$h(t) = -2\frac{M\eta}{R}[(\dot{r}^2 + r^2\dot{\Phi}^2 + \frac{M}{r}) + i(2r\dot{r}\dot{\Phi})]e^{-i2\Phi t} \tag{16}$$

3.2. **Stages of a Binary Merger.** Equation (16) is a prediction for the Gravitational Waves emitted during what is called the 'inspiral' phase of a binary merger. This is the portion where the two massive bodies orbit each other, slowly growing closer and closer together. Of the three stages of a binary merger, this is the easiest to get a concrete predictable waveform from, but it is not the stage of the merger that Gravitational Wave observatories, such as LIGO, measure, for the simple reason that GWs emitted during this period are too small. Instead, what LIGO detects is the massive GW spike that occurs when the two orbiting bodies collide, losing vast amounts of energy and a good bit of mass, and become one body. This stage is known as the Merger, and typically happens in about a tenth of a second, whereas the inspiral is known to last upwards of millions of years. Because so much energy is released in such a short amount of time, these mergers cause GWs large enough to be detectable–barely! The first GW that LIGO ever detected, GW150914, had a maximum strain of $h \approx 10^{-21}$, during its merger stage. After the merger comes the ringdown stage, in which the newly formed celestial body slowly comes to equilibrium with the gravitational field around it. Because most of the interactions during this stage happen within the body itself, there are little-no Gravitational Waves emitted from the system.

3.3. **Sidenote: Detection of the Merger.** So how do we detect these Gravitational Waves, which cause only zeptometers of change to a given length? Well, the method that LIGO (or, the Laser Interferometer Gravitational-Wave Observatory) uses is to create an incredibly sensitive interferometer. These devices are already used to detect quantities undetectable on a human scale, but in order to get to the required level of specificity, LIGO has to do a few things differently. First, in order to maximize the effect the GW will have on each arm of the interferometer, they are made vary large, exactly 4 kilometers long each. On such a large scale, a GW such as 150914 would warp each arm by about four attometers (10^{-18} meters), a much more noticeable amount than the aforementioned zeptometer (10^{-21} meters).
Second, it runs its central laser through multiple lenses and filters to ensure that the beam is made up of photons of exactly one wavelength. Third, in using these same lenses, they magnify the beam, so that the final product uses about 400 Watts of power (compare this to a typical laser pointer, which outputs only around 5 milliWatts). These things, in conjunction with several other feats of engineering and at least five decades of planning and research, allow Ligo to detect a passing Gravitational Wave as a slight dip in the intensity of its recombined beam.

4. Sources

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